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PRIMITIVE RINGS WITH INVOLUTION AND PIVOTAL MONOMIALS

BY

PAUL C. DESMARAIS† AND WALLACE S. MARTINDALE, 3rd

With the aid of ultrafilters an example is constructed of a primitive ring with involution and zero socle in which the symmetric elements S satisfy $s^2c = s$, c depending on and commuting with s (thus S satisfies the pivotal monomial x). However, by suitably restricting the definition of generalized pivotal monomial (essentially by making the variables linear) it is shown that a primitive ring with involution has nonzero socle if and only if the symmetric elements satisfy a restricted GPM.

Let $C\langle x\rangle$ be the free ring in the noncommutative indeterminates x_i, $i = 1,2,\cdots$, with C a field and $X = \{x_1, x_2, \cdots\}$. For A a C-algebra with identity let $A_0 = \{a_1, a_2, \cdots, a_p\}$ be a finite collection of C-independent elements of A. Denote the free product of A and $C\langle x\rangle$ over C by $A\langle x\rangle$. We define a monomial $\pi \in A\langle x\rangle$ to have the form

$$\pi = a_{i_0}x_{j_1}a_{i_1}x_{j_2}\cdots x_{j_k}a_{i_k}$$

where $a_{i_m} \in A_0$. Let P_π be the set of all monomials $\sigma = a_{n_0}x_{m_1}a_{n_1}\cdots x_{m_l}a_{n_l} \in A\langle x\rangle$ such that one of the following holds:

(1) $l > k$

(2) $l \leqq k$ in which case $j_t \neq m_t$ for some $t \leqq l$ or $i_t \neq n_t$ for some $t < l$.

Call M the set of all monomials in $A\langle x\rangle$ which have degree one in each variable.

DEFINITION 1. For S an additive subgroup of A,

(1) π is a (right) generalized pivotal monomial (GPM) for S if π has the above form and for each algebra homomorphism $\phi : A\langle x\rangle \to A$ with $\phi(X) \subseteq S$, $\phi(\pi) \in \phi(P_\pi)A$;

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(2) π is a restricted GPM for S if $\pi \in M$ and for each algebra homomorphism (as above) with $\phi(X) \subseteq S$, $\phi(\pi) \in \phi(P_\pi \cap M)A$;

(3) π is a pivotal monomial (restricted pivotal monomial) if π is a GPM (restricted GPM) with $A_0 = \{1\}$.

Let R be a primitive ring which we consider as an irreducible ring of endomorphisms of an additive abelian group V, so that $D = \mathrm{Hom}_R(V, V)$ is a division ring. Let C be the extended centroid of R (see, e.g., [6, p. 503]). It is easily verified [4, theor. 12, p. 453] that C is a subfield of the center of D. If R has an involution * then there is an involution $\bar{}$ defined on C. The central closure of R, defined to be $A = RC + C$, is primitive and has an involution which simultaneously extends * on R and $\bar{}$ on C, [6, theor. 4.1, p. 511]. We say R satisfies a GPM π if π is a GPM for the additive subgroup R of $RC + C$.

In Section 1 we construct an example to show that the hypothesis that the symmetric elements of a primitive ring with involution satisfy a GPM is not sufficient to guarantee a nonzero socle. In Section 2 we prove that if the symmetric elements satisfy a restricted GPM then the ring must have a nonzero socle. This generalizes a result of Amitsur [1, theor. 16, p. 225] where the ring itself is required to satisfy a GPM in order to obtain a nonzero socle.

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Our aim in this section is to present an example of a primitive ring with involution and zero socle in which the symmetric elements satisfy the pivotal monomial x.

Let V be a countably infinite dimensional vector space over K, the real numbers, with $\mathrm{Hom}_K(V, V)$ acting on V from the right. We represent $\mathrm{Hom}_K(V, V)$ as all row finite matrices over K relative to a fixed basis of V and we let U be all the elements of $\mathrm{Hom}_K(V, V)$ which have the matrix representation

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

where A is $n \times n$ and n varies. U has ordinary transpose as involution, denoted *, and is primitive (acting on V). In addition, let $\langle , \rangle$ be the standard inner product relative to the fixed basis. We note that * is the adjoint relative to $\langle , \rangle$.

REMARK 2. *If* $A \in U$ *and* $A^* = A$ *then there exists a polynomial* $p_A(x) \in K[x]$ *such that* $A = A^2 p_A(A)$.

PROOF. Let $A \in U$ and $A^* = A$. For $v \in \operatorname{Ker} A \cap \operatorname{Im} A$, $\langle v, v\rangle = \langle wA, v\rangle = \langle w, vA\rangle = \langle w, 0\rangle = 0$ and the sum $\operatorname{Ker} A + \operatorname{Im} A$ is direct. Since $\operatorname{Ker} A$ has finite codimension, $\operatorname{Ker} A \oplus \operatorname{Im} A = V$. Therefore A restricted to $\operatorname{Im} A$ is $1-1$ and there exists a polynomial $p_A(x) \in K[x]$ such that $Ap_A(A)$ is the identity on $\operatorname{Im} A$. $A^2p_A(A) = A$ on $\operatorname{Im} A$ and is zero on $\operatorname{Ker} A$. Hence $A = A^2p_A(A)$ on V.

For each $n \in N$, the natural numbers, let $U_n = U$, $N_n = N$ and denote the direct products $\Pi_{n\in N}U_n$, $\Pi_{n\in N}N_n$ as R and P respectively. R has an involution * defined componentwise by $f^*(n) = (f(n))^*$ where $f \in R$ and $f(n) \in U_n$ for each n. Since $f \in R$ is symmetric if and only if each component is symmetric Remark 2 extends to

REMARK 3. *If $f \in R$ and $f^* = f$ then $f = f^2r$ where r depends on f and $rf = fr$.*

Let F be a non-principal ultrafilter on N. Define the mapping $\mathcal{R} : R \to P$ by $\mathcal{R}(f)(n) = \operatorname{rank} f(n)$ for each $n \in N$. Form the *-ideal

$$J = \{f \in R : \exists\, k \in N, G \in F \ni \forall n \in G\ \mathcal{R}(f)(n) < k\}.$$

If $f \in J$ then we say $\mathcal{R}(f)$ is bounded on G in F. For $f, g \in R$ define the relation $<$ on R by

$$f < g \quad \text{iff} \quad \{n \in N : \mathcal{R}(f)(n) < \mathcal{R}(g)(n)\} = G \in F$$

and we say $\mathcal{R}(f) < \mathcal{R}(g)$ on G.

REMARK 4. *If $x, y \in R$ with $x < y$ then there exist $r, s \in R$ such that $x - rys \in J$.*

PROOF. $x < y$ means $\mathcal{R}(x) < \mathcal{R}(y)$ on some $G \in F$. It is well known that for each $n \in G$ there exists r_n and s_n such that $x_n = r_ny_ns_n$. For $n \notin G$ let $r_n = s_n = 0$. Then $x - rys \in J$ since $\mathcal{R}(x - rys) = 0$ on G.

We are indebted to Frank Wattenberg for the following method of construction. Let L be the set of all sequences $\lambda = (\lambda_1, \lambda_2, \cdots)$ of natural numbers such that for each $G \in F$ $\{\lambda_k\}_{k\in G}$ is unbounded. L is non-empty since F is a non-principal ultrafilter. For each $\lambda \in L$ define the idempotent e_λ by its matrix representation

$$e_\lambda(n) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is the $\lambda_n \times \lambda_n$ identity matrix. Let E be the set of all such idempotents. Finally let (x, J) denote the ideal generated by $x \in R$ and J.

REMARK 5.

(a) $E \cap J = \varnothing$

(b) *$e_\lambda < e_\gamma$ implies $e_\lambda e_\gamma - e_\lambda \in J$*

(c) *$x \in R \backslash J$ implies there exists $e_\lambda \in E$ such that $e_\lambda < x$ and $(e_\lambda, J) \subsetneq (x, J)$*

(d) *$\{e_1, e_2, \cdots, e_n\} \subseteq E$ implies there exists $e_\lambda \in E$ such that $e_\lambda < e_i$, $i = 1, 2, \cdots, n$.*

PROOF. Parts (a) and (b) follow from the definitions of E, J, $<$ and the properties of an ultrafilter. For (c) assume $x \notin J$ so that $A = \{n \in N : \mathcal{R}(x)(n) < 4\} \notin F$ and A' (the complement of A in N) $\in F$. Define the sequence λ by $\lambda_n = [\sqrt{\operatorname{rank} x_n}]$, i.e. the greatest integer less than or equal to $\sqrt{\operatorname{rank} x_n}$, if $n \in A'$ and $\lambda_n = 1$ if $n \in A$. $\lambda \in L$, otherwise $\mathcal{R}(x)$ is bounded on some element of F. By definition, $e_\lambda \in E$ and $e_\lambda < x$. By Remark 4, there exist $r, s \in R$ such that $e_\lambda - rxs \in J$. Hence $e_\lambda \in (x, J)$ and $(e_\lambda, J) \subseteq (x, J)$. However, if $x \in (e_\lambda, J)$ then $x = \sum_{i=1,p} r_i e_\lambda s_i + j$ and $\mathcal{R}(x) < p\mathcal{R}(e_\lambda) + k$ on $A' \cap G$ where $\mathcal{R}(j) < k$ on $G \in F$ and $k \in N$. Therefore $\mathcal{R}(x) < (p+k)\mathcal{R}(e_\lambda) = (p+k)[\sqrt{\operatorname{rank} x_n}]$ on $A' \cap G$. This implies $\mathcal{R}(x) < (p+k)^2$ on $A' \cap G \in F$, a contradiction since $x \notin J$. Therefore $(e_\lambda, J) \subsetneq (x, J)$.

The last part is obtained by using the properties of an ultrafilter to conclude $e_1 e_2 \cdots e_n \in E$. Then part (c) gives $e_\lambda \in E$ such that $e_\lambda < e_1 e_2 \cdots e_n$ from which it follows that $e_\lambda < e_i$, $i = 1, \cdots, n$, and the proof is complete.

Let $T = J + \sum_{\lambda \in L} (1 - e_\lambda)R$ where $(1 - e_\lambda)R = \{r - e_\lambda r : r \in R\}$ is a right ideal in R. We claim $E \cap T = \emptyset$. If not, there exists $e_\gamma \in T$ and $e_\gamma \in \sum_{i=1,n} (r_i - e_i r_i) + J$. By part (d) of Remark 5 there exists $e_\lambda \in E$ with $e_\lambda < e_\gamma$ and $e_\lambda < e_i$, $i = 1, \cdots, n$. $e_\lambda e_\gamma \in e_\lambda(\sum r_i - e_i r_i + J) \subseteq \sum(e_\lambda r_i - e_\lambda e_i r_i) + J \subseteq J$ since $e_\lambda e_i - e_\lambda \in J$ follows from $e_\lambda < e_i$ and Remark 5(b). Moreover $e_\lambda e_\gamma - e_\lambda \in J$ implies $e_\lambda \in J$, a contradiction. Hence $E \cap T = \emptyset$ and T is a proper right ideal of R containing J.

Let H be the collection of all proper right ideals, I, of R such that $T \subseteq I$ and $E \cap I = \emptyset$. Apply Zorn's lemma to this non-empty collection $(T \in H)$, which is partially ordered by inclusion, to obtain a right ideal M, maximal with respect to $T \subseteq M$ and $E \cap M = \emptyset$. M is in fact a maximal right ideal of R since if I is a right ideal which properly contains M then there exists $e_\lambda \in E$ such that $e_\lambda \in I$. But $M \subseteq I$ implies $(1 - e_\lambda)R \subseteq I$ and therefore $R = (1 - e_\lambda)R + e_\lambda R \subseteq I$. Hence M/J is a maximal right ideal in R/J and $R/M \cong (R/J)/(M/J)$ has no proper right R/J submodules.

We claim R/M is a faithful irreducible R/J module. It is enough to show R/M is faithful. If not, then there exists $x + J \neq J$ such that for each $y + M \in R/M$, $(y + M)(x + J) \subseteq M$. Hence $yx \in M$ for all $y \in R$. By Remarks 4 and 5(c) there exists $e_\lambda \in E$ with $e_\lambda < x$ and $e_\lambda + j = rxs$ for some $j \in J$ and $r, s \in R$. Letting $y = r, e_\lambda + j = rxs = yxs \in Ms \subseteq M$ and so $e_\lambda \in M$, a contradiction.

REMARK 6. *R/J is a primitive ring with involution whose symmetric elements satisfy the pivotal monomial x but which has zero socle.*

PROOF. R/J has already been shown to be right primitive. Since $J^* = J$ in R the involution in R lifts to R/J. In addition if $x + J$ is symmetric in R/J then x can be assumed to be symmetric in R, since R does not have characteristic 2. Consequently if $s + J$ is symmetric in R/J then $s = s^2r$ in R and $sr = rs$ by Remark 3. Hence $s + j = (s + J)^2(r + J)$ and $(s + J)(r + J) = (r + J)(s + J)$.

All that remains is to show that R/J has zero socle. Assume not and let Q be the socle, i.e. $0 \neq Q$ is the unique minimal ideal of R/J. Let $0 \neq x + J \in Q$. Then the ideal generated by $x + J$, $(x + J)$, is Q. Since $x \notin J$ there exists by Remark 5(c) $e_\lambda < x$ with $(e_\lambda, J) \subseteq (x, J)$ in R. Therefore in R/J we have $0 \neq (e_\lambda + J) \subsetneq (x + J) = Q$, a contradiction. Hence Q is the zero ideal.

As the preceding proof shows we have obtained an example of a primitive ring with involution whose symmetric elements satisfy $s^2r = s$ where r depends on s and commutes with s. Chacron, Herstein, and Montgomery ([2]) have recently proved the following result: a primimtive ring with involution in which $s^2p_s(s) - s$ is central for each symmetric element, where $p_s(s)$ is a polynomial in s with integral coefficients, is at most 4-dimensional over its center. Our example thus indicates limitations on attempts to generalize this result.

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In this section we let V be a right vector space over D a division ring and let C be a subfield of the center of D. We may view D as contained in $\mathrm{Hom}_C(V, V)$, which then becomes a right D-space.

DEFINITION 7. For $f \in \mathrm{Hom}_C(V, V)$, the D-rank of f is the dimension over D of the D-subspace of V spanned by Vf.

The following fundamental lemma is due to Amitsur [1, lemma 4, p. 214]:

LEMMA 8. *Let $b_1, b_2, \cdots, b_k$ be right D-independent endomorphisms of V over D. If $V_0 \subseteq V$ is a finite dimensional right D-subspace then either there exists $v \in V$ such that $vb_1, \cdots, vb_k$ are D-independent modulo V_0 or some $\Sigma_i b_i d_i \neq 0$, $d_i \in D$, has finite D-rank.*

Let R be primitive as in the introduction with $D = \mathrm{Hom}_R(V, V)$ and C, the extended centroid of R, contained in the center of D. For the remainder of this section $A = RC + C$ will denote the central closure of R.

LEMMA 9.

(1) *If a and b are elements of A such that $axb = bxa$ for all $x \in R$ then a and b are C-dependent.*

(2) $A \otimes_C D \cong AD$ *(the C-subalgebra of* $\mathrm{Hom}_C(V, V)$ *generated by A and D).*

(3) *In A, C-independence is equivalent to D-independence.*

(4) *If AD contains a nonzero finite D-ranked transformation then R contains a nonzero finite ranked transformation.*

PROOF. (1) is a special case of [6, theor. 2.1, p. 504]. The proof of (2) follows exactly as in [6, theor. 2.2, p. 504] and is included here for completeness. Define $\phi: A \otimes_c D \to AD$ by $\phi: a \otimes d \to ad$. ϕ is obviously a C-algebra homomorphism. If $\mathrm{Ker}\,\phi$ contains a nonzero element $b = \sum_{i=1,n} a_i \otimes d_i$, $a_i \in A$, $d_i \in D$, then we may assume b is of minimal "length" n. Hence the $\{a_i\}$ and the $\{d_i\}$ are C-independent collections in A and D respectively. We may assume $n > 1$ since $ad = 0$ implies $a = 0$ and $d = 0$. For each $x \in R$,

$$(a_1x \otimes 1)b - b(xa_1 \otimes 1) = \sum_{i=2,n} (a_1xa_i - a_ixa_1) \otimes d_i$$

has length less than n and is in $\mathrm{Ker}\,\phi$. The C-independence of the $\{d_i\}$ gives $a_1xa_i - a_ixa_1 = 0$ for each $x \in R$, $i = 2, \cdots, n$. By part (1), $a_i = \beta_i a_1$, $\beta_i \in C$, $i = 2, \cdots, n$, a contradiction to the C-independence of the $\{a_i\}$.

(3) is immediate from (2). For (4) assume AD contains a finite D-ranked transformation $0 \neq b = \sum_{i=1,n} a_i d_i$ which we assume to be of minimal "length" n. By the same reasoning as in the proof of (2) we obtain $a_i = \beta_i a_1$, $\beta_i \in C$, $i = 2, \cdots, n$ since $a_1xb - bxa_1$ has finite D-rank in A for all $x \in R$. Therefore letting $\beta_1 = 1$, we have $b = a_1(\sum_{i=1,n} \beta_i d_i)$ and $0 \neq \sum_{i=1,n} \beta_i d_i \in D$. Hence $0 \neq a_1 \in A = RC + C$ has finite D-rank. Actually, a_1 has finite rank over D since $a_1 \in RC + C \subseteq \mathrm{Hom}_D(V, V)$. Choose $r \in R$ such that $0 \neq ra_1 \in RC$. Then RC has a nonzero finite ranked transformation denoted $q = \sum_{i=1,n} r_i c_i$. There exist U_i, nonzero ideals of R, such that $c_i U_i \subseteq R$ for each i [5, sec. 2, p. 577]. The ideal $U = \bigcap_{i=1,n} U_i$ is nonzero since R is prime. Then $qU \subseteq R$ is a collection of finite ranked transformations in R. $qU \neq 0$, otherwise $qUC = 0$ which is a contradiction since UC is a nonzero ideal of the prime ring RC. Hence R has nonzero socle and the proof is complete.

Now we assume that R has an involution *. The concept of weak density, used by Martindale [6, pp. 508–515], can be extended as follows:

DEFINITION 10. A subset S of $R \subseteq \mathrm{Hom}_D(V, V)$ is *-weakly dense in A if it has the following property: given $v_1, v_2, \cdots, v_k$, D-independent elements of V,

$b_1, b_2, \cdots, b_m$, right D-independent elements of A and U_0 any finite dimensional D-subspace of V, then one of the following is true:

(1) $\Sigma_{i=1,m} b_i D$ contains a nonzero transformation of finite D-rank;

(2) $\Sigma_{i=1,m} b_i^* D$ contains a nonzero transformation of finite D-rank;

(3) There exists $r \in S$ such that $v_1 rb_1, v_1 rb_2, \cdots, v_1 rb_m$ are D-independent modulo U_0 and $v_i r = 0$ for $i > 1$.

LEMMA 11. *If R is a primitive ring with * then the symmetric elements, S, of R are *-weakly dense in A.*

PROOF. Let $v_1, v_2, \cdots, v_k$ be D-independent vectors in V, let $b_1, b_2, \cdots, b_m$ be right D-independent elements of A, and let U_0 be a finite dimensional D-subspace of V. Assume neither (1) nor (2) holds in Definition 10.

There exists $x \in R$ such that $v_1 x \neq 0$ and $v_i x = 0$, $i > 1$. The D-independence of $b_1^*, \cdots, b_m^*$ in A follows from Lemma 9(3), since $\Sigma b_i^* c_i = 0$, $0 \neq c_i \in C$, implies $\Sigma \bar{c}_i b_i = 0$, which contradicts the D-independence of the $\{b_i\}$. Since (2) is false, Lemma 8 can be applied to the D-independent transformations $b_1^*, \cdots, b_m^*$ to obtain $w \in V$ such that $\{wb_i^*\}$ is a D-independent set modulo the subspace generated by $v_1, v_2, \cdots, v_k$. At this point we deal with two cases:

a) If R has zero socle, pick $r \in R$ such that $v_i r = 0$, $i = 1, 2, \cdots, k$ and $wb_i^* r = wb_i^*$, $i = 1, 2, \cdots, m$. If $\{b_i^* r\}$ is a D-dependent set then $\Sigma (b_i^* r) d_i = 0$ for some $d_i \neq 0$. Hence $0 = \Sigma w(b_i^* r) d_i = \Sigma wb_i^* d_i$, a contradiction to the D-independence of the $\{wb_i^*\}$. Consequently $\{b_i^* r\}$, and hence $\{r^* b_i\}$, is a D-independent set. Since R has no nonzero finite ranked transformations, Lemma 8 assures the existence of $v \in V$ such that $\{vr^* b_i\}$ is a D-independent set modulo U_0.

b) If R has nonzero socle M, then M is an ideal of A such that $M^* = M$ and M acts densely on V. Pick $y \in M$ such that $v_i y = v_i$, $i = 1, 2, \cdots, k$ and $wb_i^* y = 0$, $i = 1, 2, \cdots, m$. Set $r = 1 - y$ and note that $v_i r = 0$, $i = 1, 2, \cdots, k$ and $wb_i^* r = wb_i^*$, $i = 1, 2, \cdots, m$. As in case a), $\{b_i^* r\}$, and hence $\{r^* b_i\}$, is a D-independent set. Since $y \in M$ and $y^* \in M$, it is clear that no nonzero D-linear combination of $(1 - y^*) b_1, \cdots, (1 - y^*) b_m$ is of finite D-rank. Again apply Lemma 8 to obtain $v \in V$ with $\{vr^* b_i\}$ a D-independent set modulo U_0.

We finish the proof by choosing $t \in R$ such that $v_1 xt = v$. Then $v_1 (xtr^* + rt^* x^*) b_j = vr^* b_j$, $j = 1, 2, \cdots, m$ and $v_i (xtr^* + rt^* x^*) = 0$ for $i > 1$. Since $xtr^* + rt^* x^* \in S$, the proof is complete.

For the remainder of this section S will denote the symmetric elements of R.

LEMMA 12. *Let $A_0 = \{a_1, a_2, \cdots, a_p\}$ be a finite C-independent subset of A and suppose neither $A_0 D$ nor $A_0^* D$ contains a nonzero finite D-ranked transfor-*

mation. Then for any sequence $a_{i_0}, a_{i_1}, \cdots$ *of elements of* A_0 *there exists* $v \in V$ *and a sequence*

$$U_0, W_0, s_1, U_1, W_1, s_2, \cdots, U_l, W_l, s_{l+1}, \cdots$$

such that

(1) $U_0 = \Sigma_{j=1,p} va_j D$, $W_0 = \Sigma_{j \neq i_0} va_j D$, $s_i \in S$

(2) $\{va_{i_0}s_1a_{i_1}s_2 \cdots a_{i_{l-1}}s_la_j\}_{j=1,p}$ *is a* D*-independent set modulo* U_{l-1}

(3) $U_l = U_{l-1} + \Sigma_{j=1,p} va_{i_0}s_1a_{i_1}s_2 \cdots a_{i_{l-1}}s_la_jD$

(4) $W_l = U_{l-1} + \Sigma_{j \neq i_l} va_{i_0}s_1a_{i_1}s_2 \cdots a_{i_{l-1}}s_la_jD$

(5) $W_ls_{l+1} = 0$.

PROOF. A_0 is a D-independent collection by Lemma 9(3). Since A_0D contains no nonzero finite D-ranked transformation, Lemma 8 yields $v \in V$ such that $\{va_j\}_{j=1,p}$ is a D–independent set of vectors in V. The D-subspace spanned by $\{va_j\}$ is denoted $U_0 = \Sigma_{j=1,p} va_jD$. Let $W_0 = \Sigma_{j \neq i_0} va_jD$. Since neither A_0D nor A_0^*D contains a nonzero finite D-ranked transformation we can apply Lemma 11 to obtain $s_1 \in S$ such that $va_{i_0}s_1a_j$ are all D-independent modulo U_0, $j = 1, \cdots, p$ and $W_0s_1 = 0$. Proceeding inductively we obtain the desired sequence where U_l and W_l are defined as in (4) and (5) and s_{i-1} is chosen, using Lemma 11, so that $va_{i_0}s_1a_{i_1} \cdots a_{i_l}s_{l+1}a_j$ are all D-independent modulo U_l, $j = 1, 2, \cdots, p$ and $W_ls_{l+1} = 0$.

THEOREM 13. *Let R be primitive with* $*$. *Suppose the symmetric elements, S, of R satisfy a restricted* GPM π. *Then A_0D or A_0^*D contains a nonzero transformation of finite D-rank.*

PROOF. Assume neither A_0D nor A_0^*D contains a nonzero transformation of finite D-rank. $\pi \in M$, so the indeterminates appearing in π are all distinct and after renumbering the subscripts on the indeterminates we may suppose that

$$\pi = a_{i_0}x_1a_{i_1}x_2 \cdots a_{i_{k-1}}x_ka_{i_k}.$$

By Lemma 12 we have $v \in V$ and symmetric elements $s_1, s_2, \cdots, s_k$ such that $va_{i_0}s_1a_{i_1} \cdots s_ka_j$ are D-independent modulo U_{k-1}, $j = 1, \cdots, p$, and $W_{l-1}s_l = 0$ for $l \leqq k$.

Consider the substitution $\phi : R\langle x \rangle \to R$ defined by $\phi(x_l) = s_l$ for $1 \leqq l \leqq k$ and $\phi(x_l) = 0$ for $l > k$. $v\phi(\pi) \neq 0$ since $v\phi(\pi)$ is one of the D-independent vectors modulo U_{k-1}. $\phi(\pi) = \Sigma_i \phi(\sigma_i)d_i$, where $d_i \in A$ depends on ϕ and $\sigma_i \in P_\pi \cap M$. We will show that $v\phi(\sigma_i) = 0$ for all i, a contradiction to $v\phi(\pi) \neq 0$.

We can assume each σ_i in the sum contains only the indeterminates x_j for $j = 1, \cdots, k$. Let

$$\sigma_i = a_{n_0} x_{m_1} a_{n_1} x_{m_2} \cdots x_{m_l} a_{n_l} \in M.$$

Each x_j in σ_i occurs at most once. We divide the argument into two cases.

(1) Assume σ_i and π first disagree (from left to right) at x_t so that $x_t \neq x_j$ for $j = m_t$. Since $\sigma_i \in M$, $j > t$ and $v\phi(a_{i_0} x_1 a_{i_1} \cdots a_{i_{t-1}} x_j) \in U_{t-1} s_j \subseteq W_t s_j \subseteq W_{j-1} s_j = 0$ by Lemma 12. Hence $v\phi(\sigma_i) = 0$.

(2) Assume σ_i and π first disagree at a_{i_t}, i.e., $a_{i_t} \neq a_{n_t}$. Then $t < k$ and σ_i does not end with a_{n_t}. $v\phi(a_{i_0} x_1 \cdots x_t a_{n_t} x_j) \in W_t s_j \subseteq W_{j-1} s_j$ where σ_i begins with $a_{i_0} x_1 a_{i_1} \cdots x_t a_{n_t} x_j$ and $j > t$. By Lemma 12, $W_{j-1} s_j = 0$ and hence $v\phi(\sigma_i) = 0$. $v\phi(\sigma_i) = 0$ in either case and we have the desired contradiction.

COROLLARY 14. *The symmetric elements of a primitive ring R with involution satisfy a restricted* GPM *if and only if R has nonzero socle.*

PROOF. Sufficiency follows from Theorem 13 and Lemma 9(4). The necessity follows by choosing any rank one idempotent e in R and verifying that $\pi = exeye$ is a restricted GPM for R. Hence π is a restricted GPM for S.

COROLLARY 15. *If the symmetric elements of a primitive ring R with involution satisfy a restricted pivotal monomial then $R \cong D_n$, the $n \times n$ matrix ring over D, a division ring.*

PROOF. Since $A_0 = \{1\}$, Theorem 13 states $A_0 D = A_0^* D = D$ contains a nonzero finite D-ranked transformation and hence $(v : D) < \infty$.

Corollary 14 is a generalization of Amitsur's Theorem 16 in [1, p. 225] while Corollary 15 is a generalization of Drazin's result ([3, theor. 4, p. 357]) that a primitive ring satisfying a pivotal monomial must be D_n. The example of Section 1 shows that requiring the symmetric elements to satisfy a pivotal monomial does not guarantee any nonzero finite ranked transformations.

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UNIVERSITY OF MASSACHUSETTS
AMHERST, MASS., U.S.A.